Problem 12

(a) Show that the function $f(x) = \ln\left(x + \sqrt{x^2+1}\right)$ is an odd function.

(b) Find the inverse function of f.

Solution

Part (a)

In order to show that the function is odd, it must be shown that $f(-x) = -f(x)$.

$$\begin{aligned} f(-x) &= \ln\left[(-x) + \sqrt{(-x)^2+1}\right] \\ &= \ln\left(-x + \sqrt{x^2+1}\right) \\ &= \ln\left[\left(-x + \sqrt{x^2+1}\right) \times \frac{-x - \sqrt{x^2+1}}{-x - \sqrt{x^2+1}}\right] \\ &= \ln \frac{\left(-x + \sqrt{x^2+1}\right)\left(-x - \sqrt{x^2+1}\right)}{-x - \sqrt{x^2+1}} \\ &= \ln \frac{(-x)^2 - (x^2+1)}{-x - \sqrt{x^2+1}} \\ &= \ln \frac{x^2 - x^2 - 1}{-x - \sqrt{x^2+1}} \\ &= \ln \frac{-1}{-x - \sqrt{x^2+1}} \\ &= \ln \frac{1}{x + \sqrt{x^2+1}} \\ &= \ln\left(x + \sqrt{x^2+1}\right)^{-1} \\ &= -\ln\left(x + \sqrt{x^2+1}\right) \\ &= -f(x) \end{aligned}$$

Therefore, the function $f(x) = \ln\left(x + \sqrt{x^2+1}\right)$ is an odd function.

Part (b)

In order to find the inverse of

$$f(x) = \ln\left(x + \sqrt{x^2+1}\right),$$

replace $f(x)$ with x, and replace x with y.

$$x = \ln\left(y + \sqrt{y^2+1}\right)$$

Then solve for y. Start by exponentiating both sides.

$$e^x = e^{\ln\left(y+\sqrt{y^2+1}\right)}$$

$$e^x = y + \sqrt{y^2+1}$$

Bring y to the left side.

$$e^x - y = \sqrt{y^2+1}$$

Square both sides.

$$(e^x - y)^2 = y^2 + 1$$

Expand the left side.

$$e^{2x} - 2ye^x + y^2 = y^2 + 1$$

Subtract both sides by y^2.

$$e^{2x} - 2ye^x = 1$$

Bring 1 to the left, and bring $2ye^x$ to the right.

$$e^{2x} - 1 = 2ye^x$$

Divide both sides by $2e^x$.

$$\frac{e^{2x}-1}{2e^x} = y$$

Therefore, the inverse function is

$$f^{-1}(x) = \frac{e^{2x}-1}{2e^x}.$$